J. Fluid Mech. (1971), vol. 49, part 1, pp. 21–31
Printed in Great Britain

Hydromagnetic instability of a free shear layer at small magnetic Reynolds numbers

By KANEFUSA GOTOH
Research Institute for Mathematical Sciences,
University of Kyoto, Kyoto

(Received 25 June 1970)

The effect of a uniform and parallel magnetic field upon the stability of a free shear layer of an electrically conducting fluid is investigated. The equations of the velocity and the magnetic disturbances are solved numerically and it is shown that the flow is stabilized with increasing magnetic field. When the magnetic field is expressed in terms of the parameter N ($= M^2/R^2$), where M is the Hartmann number and R is the Reynolds number, the lowest critical Reynolds number is caused by the two-dimensional disturbances. So long as $0 \leqslant N \leqslant 0{\cdot}0092$ the flow is unstable at all R. For $0{\cdot}0092 < N \leqslant 0{\cdot}0233$ the flow is unstable at $0 < R < R_{uc}$, where R_{uc} decreases as N increases. For $0{\cdot}0233 < N < 0{\cdot}0295$ the flow is unstable at $R_{lc} < R < R_{uc}$, where R_{lc} increases with N. Lastly for $N > 0{\cdot}0295$ the flow is stable at all R. When the magnetic field is measured by M, the lowest critical Reynolds number is still due to the two-dimensional disturbances provided $0 \leqslant M \leqslant 0{\cdot}52$, and R_c is given by the corresponding R_{lc}. For $M > 0{\cdot}52$, R_c is expressed as $R_c = 5{\cdot}8M$, and the responsible disturbance is the three-dimensional one which propagates at angle $\cos^{-1}(0{\cdot}52/M)$ to the direction of the basic flow.

1. Introduction

It is well known that the magnetic field imposed upon laminar flows of electrically conducting fluid has a tendency to stabilize the flows. This tendency has in fact been confirmed for flows such as channel flow (Stuart 1954; Hains 1965) and boundary-layer flow (Abas 1968) when the magnetic field is uniform and parallel to the flow and the magnetic Reynolds number is small. In these cases the critical Reynolds number, which is already quite large without the magnetic field, is increased with increasing intensity of the magnetic field. On the other hand, the situation is not as clear in unbounded flows such as jet, wake, and free shear layer flows between two parallel streams. Without a magnetic field these flows are much more unstable compared with bounded flows, and above all the free shear layer is unstable at all Reynolds numbers. Thus if the stabilizing effect of the magnetic field is also to exist for unbounded flows, it should change drastically the stability characteristics of the flows. So far, however, no decisive result seems to have been obtained for this problem.

The stability of a free shear layer in the presence of a uniform and parallel magnetic field has so far been dealt with only for the cases of infinitesimal Reynolds numbers (Abas 1969) and very large Reynolds numbers (Gotoh & Numata 1969). With these investigations, however, the central problem of how the critical Reynolds number is changed with the magnetic field has not been solved and the matter has been left for conjecture (Abas 1969). In the present paper we shall deal with the same problem using a numerical method and thus the stability characteristics will be examined for all values of the Reynolds number R. As a result it will be shown that the flow is in fact stabilized by the magnetic field.

2. Formulation of the problem

Denote the velocity of a plane parallel flow by $(U(y), 0, 0)$ and the uniform parallel magnetic field by $(H, 0, 0)$, where the x axis of the Cartesian co-ordinates (x, y, z) is taken along the direction of the steady flow. The instability of this hydromagnetic flow due to infinitesimal disturbances is investigated. In view of the independence of the basic fields on x, z and time t, we may decompose the disturbance of the velocity $\mathbf{u}'$ and that of the magnetic field $\mathbf{h}'$ into normal modes:

$$\begin{Bmatrix}\mathbf{u}'(x,y,z;t)\\ \mathbf{h}'(x,y,z;t)\end{Bmatrix} = \begin{Bmatrix}\hat{\mathbf{u}}(y)\\ \hat{\mathbf{h}}(y)\end{Bmatrix} \exp\left[i(\alpha x+\beta z-\alpha c t)\right],$$

where $\alpha(>0)$ and β (real) denote the wave-number in the x and z directions, and c $(=c_r+ic_i)$ the complex phase velocity of the disturbance. According as c_i takes positive, zero or negative values, the disturbance is amplified, neutral or damped, respectively.

When the equations for disturbances $\hat{\mathbf{u}}$ and $\hat{\mathbf{h}}$ are made non-dimensional there appear the following parameters:

$$\begin{aligned} R &= LU_0/\nu, \text{ Reynolds number:}\\ R_m &= 4\pi\sigma\mu U_0 L, \text{ magnetic Reynolds number;}\\ M &= (\sigma/\nu\rho)^{\frac{1}{2}}\mu LH, \text{ Hartmann number;}\end{aligned}$$

where U_0 and L are the representative velocity and length, and ν, σ, μ and ρ denote the kinematic viscosity, the electric conductivity, the magnetic permeability and the density of the fluid, respectively. If we consider flows of a fixed kind of fluid, R_m $(=4\pi\sigma\nu\mu R)$ is not independent of R.

The boundary-value problem for $\hat{\mathbf{u}}$ and $\hat{\mathbf{h}}$ leads to a relationship between eigenvalues of c, α, β, R and M, from which we can determine c as a function of the rest. It may easily be seen that these parameters appear in the relation only in the form, $\kappa = (\alpha^2+\beta^2)^{\frac{1}{2}}$, αR and $N = M^2/R^2$, where N is another parameter representing the magnetic field.† Thus if we take R and N as independent parameters, c_i may be expressed as

$$c_i = F(\kappa, \alpha R, N),$$

where F is a function to be determined by the eigenvalue relation. For the neutral disturbances we have

$$F(\kappa, \alpha R, N) = 0,$$

or alternatively,

$$\alpha R = G(\kappa, N). \tag{2.1}$$

† N is equivalent to Q/R used by Abas (1969) and N/R used by Gotoh & Numata (1969).

It may easily be shown from (2.1) that Squire's theorem concerning the critical Reynolds number of the non-magnetic flows is equally valid in this case. Let us denote the critical Reynolds number for three-dimensional disturbances (β/α = non-zero constant) by $\{R_c(N)\}_3$ and the two-dimensional one ($\beta = 0$) by $\{R_c(N)\}_2$. Then from (2.1) we have

$$\begin{aligned}\{R_c(N)\}_3 &= \min_{\kappa>0}[(1/\alpha)\,G(\kappa,N)] = (\kappa/\alpha)\min_{\kappa>0}[(1/\kappa)\,G(\kappa,N)]\\ &= (\kappa/\alpha)\{R_c(N)\}_2 > \{R_c(N)\}_2,\end{aligned} \tag{2.2}$$

where $\{R_c(N)\}_2 = \min\limits_{\alpha>0}[(1/\alpha)\,G(\alpha,N)]$ by definition. Thus the two-dimensional disturbance gives the lowest critical Reynolds number when N is fixed.

In practice, however, the magnetic field and the flow velocity are controlled separately, and therefore a more natural choice of independent parameters may be R and M rather than R and N. The inequality (2.2) does not apply for fixed M, but the above argument is easily modified to cover this case. Equation (2.1) may be rewritten as

$$\alpha R = G(\kappa, \alpha M). \tag{2.3}$$

If we denote the critical Reynolds number for fixed M by $R_c(M)$ and proceed similarly to the above, we have

$$\{R_c(M)\}_3 = (\kappa/\alpha)\{R_c(\alpha M/\kappa)\}_2. \tag{2.4}$$

Thus, although Squire's theorem does not necessarily hold for fixed M we can easily calculate the critical Reynolds number for three-dimensional disturbances from the two-dimensional one through (2.4) provided the latter is known for all values of M. The same result has already been obtained by Hunt (1966) for the case of small magnetic Reynolds numbers.

In the following we shall first investigate the stability of a free shear layer of the velocity profile $U(y) = \tanh y$ against two-dimensional disturbances and later deal with three-dimensional disturbances. If we consider conventional electrically conducting fluids such as mercury, R_m/R is of order of 10^{-7}, and hence we can safely assume that $R_m \ll 1$ while R is arbitrary so long as these fluids are concerned. The equations governing two-dimensional disturbances are reduced, when R_m is small, to the following single equation

$$(D^2-\alpha^2)^2\phi - i\alpha R[(U-c)(D^2-\alpha^2) - D^2U + i\alpha RN]\phi = 0, \tag{2.5}$$

where $D = d/dy$, $\phi(y) = \hat{u}_y(y)/i\alpha$. Since the disturbances should vanish at infinity, $\phi(y)$ must satisfy the boundary condition

$$\phi(\infty) = \phi(-\infty) = 0. \tag{2.6}$$

The equation (2.5) together with the boundary condition (2.6) constitutes an eigenvalue problem for c, α, R and N.

3. Eigenvalue problem

Equation (2.5) permits four particular solutions. Let us denote two of them which vanish at $y = \infty$ by ϕ_1 and ϕ_2. Then the solution of (2.5) and (2.6) may be expressed as

$$\phi = C_1\phi_1 + C_2\phi_2, \tag{3.1}$$

where C's are complex constants to be determined so as to satisfy the condition $\phi(-\infty) = 0$.

Since we are interested only in neutral disturbances, we may put $c = 0$.† In the case of an antisymmetric velocity profile the condition $c = 0$ requires that the solution of (2.5) and (2.6) has Hermitian symmetry with respect to y. So we can replace the condition $\phi(-\infty) = 0$ by the symmetry condition at $y = 0$:

$$D^k\phi(0) = (-1)^k D^k\overline{\phi}(0) \quad (k = 0, 1, 2, \text{ and } 3), \tag{3.2}$$

where $\overline{\phi}$ denotes the complex conjugate of ϕ. The conditions for all higher derivatives are then automatically satisfied through (3.2) and (2.5). Substituting (3.1) into (3.2), we have the following condition for all C's not to vanish

$$\begin{vmatrix} \phi_1(0) & \overline{\phi}_1(0) & \phi_2(0) & \overline{\phi}_2(0) \\ D\phi_1(0) & -D\overline{\phi}_1(0) & D\phi_2(0) & -D\overline{\phi}_2(0) \\ D^2\phi_1(0) & D^2\overline{\phi}_1(0) & D^2\phi_2(0) & D^2\overline{\phi}_2(0) \\ D^3\phi_1(0) & -D^3\overline{\phi}_1(0) & D^3\phi_2(0) & -D^3\overline{\phi}_2(0) \end{vmatrix} = 0. \tag{3.3}$$

For numerical integration it is convenient to transform (2.5) into a set of first-order equations. For this purpose we transform the independent variable into

$$z = U(y) = \tanh y, \tag{3.4}$$

and introduce dependent variables

$$f_{jn} = (-\lambda_j)^{-n}(1+i)\, D^n\phi_j/\phi_j \quad (j = 1, 2; n = 1, 2, 3), \tag{3.5}$$

where $\quad \lambda_{1,2}^2 = \alpha^2 + \tfrac{1}{2} i\alpha R\,[1 \pm (1+4N)^{\frac{1}{2}}], \quad \mathrm{Re}\,[\lambda_{1,2}] > 0,$

have been introduced in order to keep the magnitude of f_{jn} at $z = 1$ within the order of unity. Then, (2.5) may be expressed as

$$\left.\begin{aligned} \frac{df_{j1}}{dz} &= -\frac{\lambda_j}{1-z^2}\left(f_{j2} - \frac{f_{j1}^2}{1+i}\right), \\ \frac{df_{j2}}{dz} &= -\frac{\lambda_j}{1-z^2}\left(f_{j3} - \frac{f_{j1}f_{j2}}{1+i}\right), \\ \frac{df_{j3}}{dz} &= \frac{1}{1-z^2}\left[\frac{\lambda_j}{1+i} f_{j1}f_{j3} - \frac{2\alpha^2}{\lambda_j} f_{j2} + \frac{\alpha^4}{\lambda_j^3}(1+i)\right. \\ &\qquad \left. + \frac{i\alpha R(1+i)}{\lambda_j^3}\left\{-2z(1-z^2) + \alpha^2 z - i\alpha RN - \frac{\lambda_j^2}{1+i} z f_{j2}\right\}\right], \end{aligned}\right\} \tag{3.6}$$

where use has been made of the fact that $U = \tanh y$ and that $c = 0$. The boundary conditions for f_{jn} are obtained by substitution of the asymptotic forms of $\phi_j (j = 1, 2)$ for $y \gg 1$ into (3.5) and (3.6) as follows: at $z = 1$,

$$f_{jn} = 1 + i, \tag{3.7}$$

† In general the neutrality of disturbance requires simply that $c_i = 0$, but here we take the condition $c = 0$ as usually done. If we assume the uniqueness of solution of (2.5) and (2.6) for antisymmetric velocity profiles, then $c = 0$ follows exactly as in non-magnetic case (see Tatsumi & Gotoh 1960).

$$\left.\begin{aligned} \frac{df_{j1}}{dz} &= \frac{i\alpha R(1+i)(\lambda_j^2-\alpha^2-4)}{2\lambda_j(1+\lambda_j)[2(2+2\lambda_j+\lambda_j^2-\alpha^2)-i\alpha R]}, \\ \frac{df_{j2}}{dz} &= 2\left(1+\frac{1}{\lambda_j}\right)\frac{df_{j1}}{dz}, \\ \frac{df_{j3}}{dz} &= \left(3+\frac{6}{\lambda_j}+\frac{4}{\lambda_j^2}\right)\frac{df_{j1}}{dz}. \end{aligned}\right\} \tag{3.8}$$

The eigenvalue relation (3.3) may be rewritten in terms of f_{jn} as follows:

$$E(\alpha,R,N) = \begin{vmatrix} 1 & -1 & 1 & -1 \\ F_{11} & \overline{F}_{11} & F_{21} & \overline{F}_{21} \\ F_{12} & -\overline{F}_{12} & F_{22} & -\overline{F}_{22} \\ F_{13} & \overline{F}_{13} & F_{23} & \overline{F}_{23} \end{vmatrix} = 0, \tag{3·9}$$

where $$F_{jn} = (-\lambda_j)^n f_{jn}(0)/[(1+i)|\lambda_1\lambda_2|^{\frac{1}{2}n}],$$

the factor $|\lambda_1\lambda_2|^{\frac{1}{2}n}$ being introduced to improve accuracy of numerical calculation. Equation (3.9) is the eigenvalue equation which determines the relationship between α, R and N for the neutral disturbance.

4. Results and discussions

4.1. *Two-dimensional disturbances*

As noted in §2, the two-dimensional disturbances give the lowest critical Reynolds number when N is fixed. Thus we shall first work out the curves of neutral stability against two-dimensional disturbances for various constant values of N. We solve the eigenvalue equation (3.9) numerically using an iteration method as described below.

In the first step, take a trial value, α_0 say, of α for a given pair of R and N, and integrate (3.6) numerically starting from the point $z = 1$ and find $f_{jn}(0)$. Substitution of $f_{jn}(0)$ into the determinant of (3.9) gives $E(\alpha_0, R, N)$, which will be denoted as $E(\alpha_0)$ for brevity. In general $E(\alpha_0) \neq 0$ and then we calculate $E(\alpha_0+\Delta\alpha_0)$ for a certain increment $\Delta\alpha_0$. Now there are three possibilities. (i) If $E(\alpha_0)E(\alpha_0+\Delta\alpha_0) > 0$ and $|E(\alpha_0+\Delta\alpha_0)| > |E(\alpha_0)|$, calculate $E(\alpha_0+k\Delta\alpha_0)$ for $k = 2, 3, \ldots$ until we first obtain $E(\alpha_0+n\Delta\alpha_0)\ E(\alpha_0+(n+1)\Delta\alpha_0) < 0$ for a certain n. Put $\alpha_0+n\Delta\alpha_0 = \alpha_1$. (ii) If $E(\alpha_0)E(\alpha_0+\Delta\alpha_0) > 0$ and $|E(\alpha_0+\Delta\alpha_0)| < |E(\alpha_0)|$, calculate $E(\alpha_0-k\Delta\alpha_0)$ for $k = 1, 2, \ldots$ until we first have $E(\alpha_0-n\Delta\alpha_0)E(\alpha_0-(n+1)\Delta\alpha_0) < 0$ for a certain n. Put $\alpha_0-n\Delta\alpha_0 = \alpha_1$. (iii) If $E(\alpha_0)E(\alpha_0+\Delta\alpha_0) < 0$, put $\alpha_0 = \alpha_1$. Thus we obtain the first approximation $|E(\alpha_1)|$, which must be much smaller than $|E(\alpha_0)|$.

In the second step, take α_1 and $\Delta\alpha_1 = 10^{-1}\Delta\alpha_0$ as the trial α and its increment respectively and proceed as before until we find an α_2 satisfying the respective inequalities in (i), (ii) and (iii). Thus we find the second approximation $|E(\alpha_2)|$. Proceeding similarly to the third and further steps we obtain $|E(\alpha_n)| \leqslant 10^{-3}$, and take α_n as the eigenvalue of α for the given pair of R and N. The eigenvalues of α for different values of R and N are calculated in the same manner, and we obtain the numerical results as tabulated in table 1.

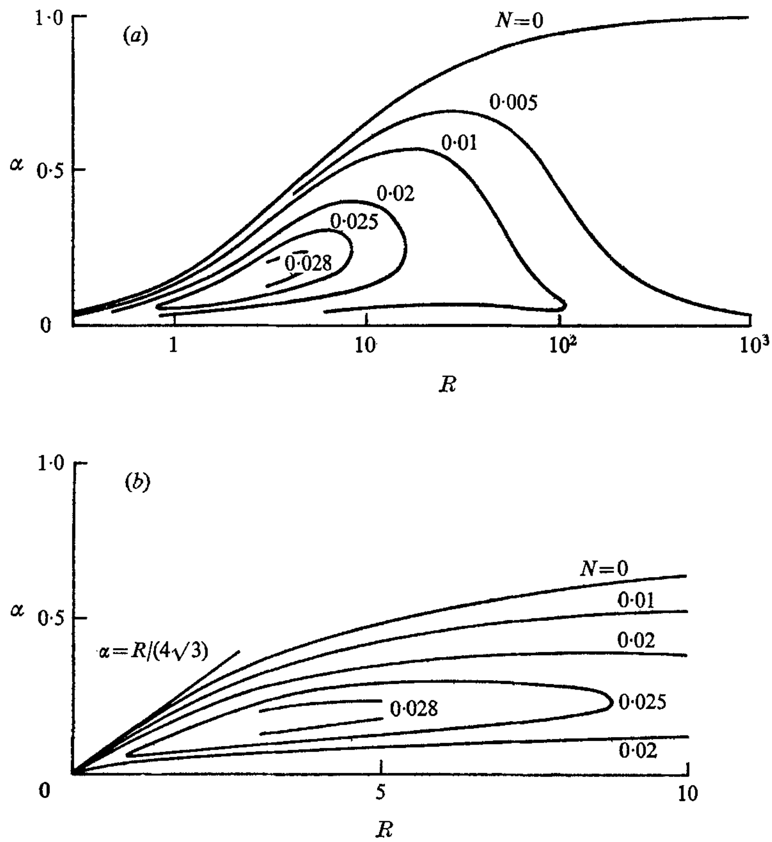

FIGURE 1. Distribution of the neutral curves of two-dimensional disturbances for various values of N.

$N = 0$		$N = 0{\cdot}005$	
R	α	R	α
0·5	0·0734	5·0	0·4648
1·0	0·1470	10·0	0·6031
2·0	0·2731	15·0	0·6609
3·0	0·3671	20·0	0·6873
4·0	0·4380	25·0	0·6978
5·0	0·4933	30·0	0·6990
7·0	0·5743	40·0	0·6851
9·0	0·6315	50·0	0·6584
10·0	0·6543	60·0	0·6242
15·0	0·7346	70·0	0·5852
20·0	0·7834	80·0	0·5435
25·0	0·8167	90·0	0·5000
30·0	0·8409	100·0	0·4565
		150·0	0·2842
		200·0	0·1952

$N = 0{\cdot}01$

R	α Upper branch	α Lower branch
0·5	0·0632	—
1·0	0·1285	—
2·0	0·2426	—
3·0	0·3265	—
4·0	0·3876	—
5·0	0·4333	—
7·0	0·4953	—
9·0	0·5330	—
10·0	0·5461	—
15·0	0·5769	—
20·0	0·5748	—
30·0	0·5243	0·06693
40·0	0·4423	0·06556
50·0	0·3481	0·06341
70·0	—	0·05918
80·0	0·1475	—
100·0	—	0·05678
105·0	0·0793	0·05784
110·0	0·0653	0·06360
115·0	No solution	

$N = 0{\cdot}0275$

R	α Upper branch	α Lower branch
3·0	0·2113	—
4·0	0·2450	0·1442
5·0	0·2565	0·1697

$N = 0{\cdot}028$

R	α Upper branch	α Lower branch
3·0	0·2038	0·1283
4·0	0·2355	0·1524
5·0	0·2415	0·1832

$N = 0{\cdot}02$

R	α Upper branch	α Lower branch
0·5	0·0485	—
1·0	0·1030	—
2·0	0·2027	—
3·0	0·2745	0·0683
4·0	0·3234	0·0808
5·0	0·3563	0·0913
7·0	0·3911	0·1090
9·0	0·3990	0·1248
10·0	0·3956	0·1327
15·0	0·3150	0·1890
16·0	0·2716	0·2207
17·0	No solution	

$N = 0{\cdot}025$

R	α Upper branch	α Lower branch
0·5	No solution	
0·8	0·0566	0·0565
1·0	0·0791	0·0524
1·5	0·1298	0·0659
2·0	0·1734	0·0779
3·0	0·2379	0·0983
4·0	0·2784	0·1159
5·0	0·3012	0·1324
6·0	0·3104	0·1492
7·0	0·3084	0·1680
8·0	0·2928	0·1925
8·4	0·2802	0·2068
8·6	0·2709	0·2168
8·8	0·2355	0·2342
9·0	No solution	

TABLE 1. Wave-numbers of the neutral disturbance for various values of N and R

The neutral curves in the (α, R) plane are depicted in figure 1 (a), and their behaviour for small values of R is shown in figure 1 (b). It may easily be seen from these figures that the unstable region of the (α, R) plane decreases monotonically with increasing N. There are four stages of the neutral curves for different N.

(i) For $0 \leqslant N < 0{\cdot}0092$ the neutral curves extend from $R = 0$ to $R = \infty$, so that the flow remains unstable at all Reynolds numbers. For extremely small values of α and R the behaviour of the neutral curves is in accordance with that derived from the values of R/α given by Abas (1969).

	$\alpha = 0$	
N	αR	αR
0·003	—	1·878
0·004	—	2·127
0·005	33·49	2·416
0·007	18·68	3·240
0·008	13·68	3·945
0·0085	11·36	4·508
0·0090	8·739	5·579
0·0095	No solution	

TABLE 2. Eigenvalues of αR in the limit: $\alpha \to 0$

In order to obtain the eigenvalue relation for very large values of R and small α, we put $\alpha = 0$ and search the eigenvalue of αR for various constant values of N. The result is tabulated in table 2. The relation between αR and N is as shown in figure 2, which gives two values of αR for a given N, or in other words, two

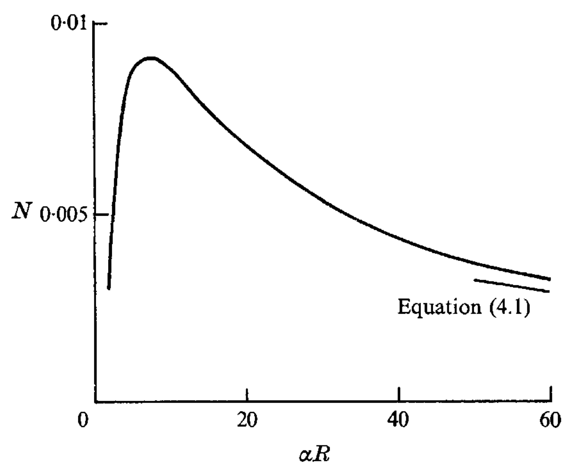

FIGURE 2. Eigenvalue of αR in the limit: $\alpha \to 0$.

asymptotic branches of the neutral curves of the form $R \propto \alpha^{-1}$. For large values of αR the upper branch is well approximated by the formula

$$N = \frac{1}{4\alpha R}\left(1 - \frac{18}{\alpha R}\right), \tag{4.1}$$

which was given by Gotoh & Numata (1969). Figure 2 shows that two branches coalesce to one at $N = 0{\cdot}0092$, N_1 say, so that there exists no asymptotic branch of this kind for $N > N_1$.

(ii) For $N_1 \leqslant N < 0{\cdot}0233$ the neutral curve describes a loop on the higher Reynolds number side, and therefore the flow is unstable in the range $0 < R < R_{uc}$, where R_{uc} may be called the upper critical Reynolds number. R_{uc} decreases monotonically from infinity at $N = N_1$ with increasing N. Two asymptotic branches of the neutral curve at very small R approach to each other as N increases and eventually coalesce at $N = 0{\cdot}0233$, $= N_2$ say. Abas (1969) conjectured that $N > N_2$ would correspond to the state of complete stability, but as will be shown below the neutral curve still exists for $N > N_2$.

(iii) For $N_2 \leqslant N < 0{\cdot}0295$ the neutral curve describes a closed contour, and hence the flow is unstable in the range $R_{lc} < R < R_{uc}$, where R_{lc} may be called the lower critical Reynolds number. R_{lc} increases monotonically from zero at $N = N_2$ with increasing N. The change of the critical Reynolds number R_c, including both R_{lc} and R_{uc}, with N is shown graphically in figure 3 which has been produced from

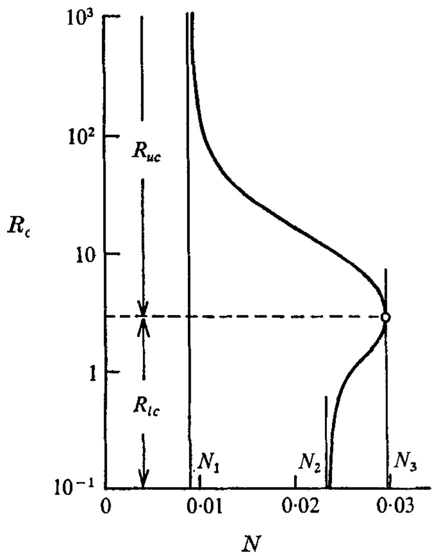

FIGURE 3. Critical Reynolds numbers for two-dimensional disturbances when N is fixed as a magnetic parameter.

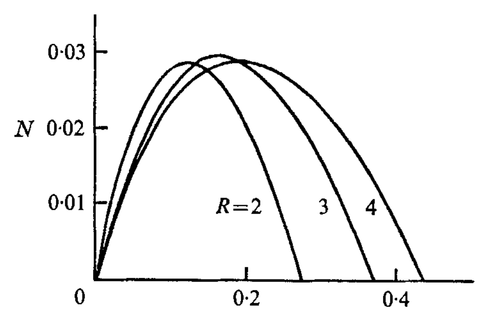

FIGURE 4. Distribution of N at various values of R. N takes its maximum value 0·0295 on the curve of $R = 3{\cdot}0$.

data in figures 1 (*a*) and (*b*). At the value of $N = 0{\cdot}0295$, $= N_3$ say, R_{lc} and R_{uc} become identical, that is the neutral curve vanishes.† Figure 4 shows that the neutral curve vanishes at the point $R = 3{\cdot}0$ and $\alpha = 0{\cdot}16$.

(iv) For $N > N_3$ there is no neutral curve, and the flow is stable at all Reynolds numbers. Thus the minimum value of N which gives complete stability does not correspond to $N_2 = 0{\cdot}0233$ as conjectured by Abas but to $N_3 = 0{\cdot}0295$.

4.2. *Three-dimensional disturbances*

As mentioned in §2, when we take the Hartmann number M as a magnetic parameter in place of N, two-dimensional disturbances do not necessarily give the lowest critical Reynolds number. We can, however, easily calculate the critical

† It can be proved that there is an upper bound of N for the amplified disturbance to exist (Gotoh 1971).

Reynolds number due to three-dimensional disturbances from the two-dimensional one using the equation

$$\{R_c(M)\}_3 = \frac{\kappa}{\alpha}\left\{R_c\left(\frac{\alpha}{\kappa}M\right)\right\}_2. \tag{2.4}$$

First we calculate $\{R_c(M)\}_2$ from $\{R_c(N)\}_2$ given in figure 3 through the relation $N = M^2/R^2$. The resulting $\{R_c(M)\}_2$ is shown graphically in figure 5.

It may be seen that the ratio $\{R_c(M)\}_2/M$ decreases as M increases in the range $0 < M < 0{\cdot}52 = N_3^{\frac{1}{2}}\{R_c(N_3)\}_2$, while it increases with M for $M > 0{\cdot}52$. Hence, for $0 < M < 0{\cdot}52$,

$$\left\{R_c\left(\frac{\alpha}{\kappa}M\right)\right\}_2 \bigg/ \left(\frac{\alpha}{\kappa}M\right) > \{R_c(M)\}_2/M,$$

so that from (2.4) we have

$$\{R_c(M)\}_3 > \{R_c(M)\}_2. \tag{4.2}$$

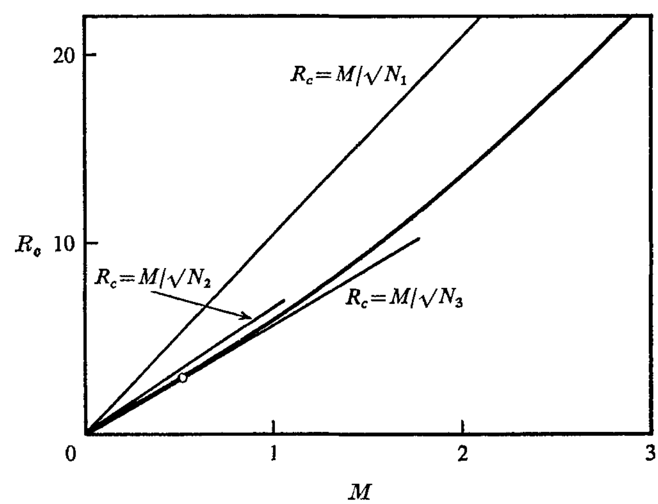

FIGURE 5. Critical Reynolds numbers for two-dimensional disturbances when M is fixed as a magnetic parameter.

Thus the lowest critical Reynolds number is due to the two-dimensional disturbance, or in other words, Squire's theorem is valid in the range $0 \leqslant M \leqslant 0{\cdot}52$.

For $M > 0{\cdot}52$, on the other hand, the same argument leads to

$$\begin{aligned}\min_{\kappa/\alpha}\{R_c(M)\}_3 &= \min_{\kappa/\alpha}\frac{\kappa}{\alpha}\left\{R_c\left(\frac{\alpha}{\kappa}M\right)\right\}_2 \\ &\geqslant \frac{M}{0{\cdot}52}\{R_c(0{\cdot}52)\}_2 \\ &= 5{\cdot}8M. \end{aligned}\tag{4.3}$$

Thus the lowest critical Reynolds number for all possible three-dimensional disturbances is given by (4.3), and the responsible disturbance is the one which propagates in the direction $\theta = \cos^{-1}(0{\cdot}52/M)$.

The author wishes to express his cordial thanks to Professor T. Tatsumi for his helpful advice and discussions in revising the original manuscript.

REFERENCES

ABAS, S. 1968 *J. Fluid Mech.* **32**, 721.
ABAS, S. 1969 *J. Fluid Mech.* **38**, 243.
GOTOH, K. 1971 *J. Phys. Soc. Japan*, **31**, 317.
GOTOH, K. & NUMATA, T. 1969 *J. Phys. Soc. Japan*, **27**, 764.
HAINS, F. D. 1965 *Phys. Fluids*, **8**, 2014.
HUNT, J. C. R. 1966 *Proc. Roy. Soc.* A **293**, 342.
STUART, J. T. 1954 *Proc. Roy. Soc.* A **221**, 189.
TATSUMI, T. & GOTOH, K. 1960 *J. Fluid Mech.* **7**, 433.